Conductivity of One-Dimensional Interacting Fermions*

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Using an exactly soluble model, the decay rate of a current-carrying state of one-dimensional fermions is calculated in the presence of random scatterers at finite temperature and the dc conductivity thereby inferred. For interacting fermions it is modified by a factor $(T/T+\theta)^g$, where g is a positive (negative) coupling constant for repulsive (attractive) two-body forces. While the conductivity could be greatly enhanced for $g<0$ and $T \ll \theta$, one-dimensional superconductivity appears ruled out at any finite temperature.

The conductivity $\sigma(T)$ of a gas of interacting one-dimensional (1D) fermions at finite temperature has been obtained in the presence of random scatterers. The main results can be summarized in the formula

$$\sigma(T) = [T/(T+\theta)]^g \sigma_0 \tag{1}$$

in which σ_0 is the dc conductivity of a noninteracting Fermi gas ($g=0$), a quantity which is inversely proportional to the density of scattering centers and to the scattering strength of each, and which is finite at $T=0$ and slowly varying with temperature. The exponent g characterizes the strength of the two-body interactions, and is positive for repulsive forces and negative for attractive forces. θ is a constant related to the range of the forces, expressed in degrees kelvin.

Equation (1), which is asymptotically exact in the weak-coupling limit $|g| \ll 1$, indicates that even for weakly repulsive two-body forces the electrical conductivity vanishes at $T=0$, and that, conversely, for even weakly attractive two-body forces the conductivity increases without limit as $T \to 0$. These results agree at $T=0$ with my earlier analysis of the ground-state conductivity[1] and are also compatible with an independent study of Green functions and correlation functions at finite temperature[2] in which impurities were not explicitly considered. Interest in the topic has recently intensified as a result of interesting experimental studies of pseudo-1D structures, such as tetrathiofulvalinium tetracyanoquinodimethane (TTF-TCNQ), by Heeger and co-workers,[3] generating some controversy,[4] but also rekindling the hope of discovering high-temperature superconductivity, perhaps in 1D manifolds. The formula Eq. (1) precludes 1D superconductivity on purely theoretical grounds but allows for substantial enhancement of the conductivity over that for noninteracting particles, as seems to be the case experimentally.[3]

The basis of the present work is an extension of Fermi's "Golden Rule" to finite temperature, viz.,

$$dj/dt=(2v_0/\hbar^2)\int_{-\infty}^{+\infty}dt\langle[H^+(0),H^-(t)]\rangle. \tag{2}$$

In this formula, the angular brackets signify thermal and configurational averages. $H^{\pm}$ represent the forward/backward scattering matrix elements, chosen to be

$$\begin{aligned}H^+&=\int dx\, W^*(x)\psi_1^\dagger(x)\psi_2(x)\equiv\int dx\, W^*(x)H^+(x),\\ H^-&=\int dx\, W(x)\psi_2^\dagger(x)\psi_1(x)\equiv\int dx\, W(x)H^-(x),\end{aligned} \tag{3}$$

with $W(x)$ a random traceless scattering potential. The time dependence of operators is given by

$$H^{\pm}(x,t)=e^{iHt/\hbar}H^{\pm}(x)e^{-iHt/\hbar}, \tag{4}$$

where the fermion Hamiltonian is given as

$$H=\hbar v_0\sum k(n_{1k}-n_{2k})+(\lambda/L)\sum U(p)|[\rho_1(p)+\rho_2(p)]|^2 \tag{5}$$

and the current operator is

$$j_{\rm op}=v_0\sum(n_{1k}-n_{2k}). \tag{6}$$

The fermion Hamiltonian (5) was originally proposed by Luttinger,[5] solved by Lieb and the present author,[6] and is discussed in our book on one-dimension,[7] to which the reader is referred for background information and algebraic details. In addition, a very useful representation of the wave operators $\psi(x)$ as exponentials of the density-fluctuation operators $\rho(p)$ was recently discovered, simultaneously and independently, by Luther and Peschel[2] and by the present author.[1] This representation permits the explicit and exact evaluation of operators such as (4) and thermal averages such as (2), which could not otherwise be performed for interacting particles.

I briefly summarize the steps leading to Eq. (1), and conclude this work by comparing it to calculations[8] in which the BCS theory[9] is applied to the study of the possible superconductivity of 1D electrons.

First, it is most convenient to evaluate traces such as occur in Eq. (2) in the representation where H is diagonal. The appropriate unitary transformation, $O\to\exp(+iS)O\exp(-iS)$, is given by

$$S=(2\pi i/L)\sum_{\text{all }p}p^{-1}\varphi(p)\rho_1(p)\rho_2(-p). \tag{7}$$

We recall the commutation relations among the ρ's[1,6,7]:

$$[\rho_i(p),\rho_j(-p)]=(-1)^i\delta_{ij}pL/2\pi \tag{8}$$

with $i,j=1$ for the right-going, and 2 for the left-going particles. The correct value of $\varphi(p)$ is[1]

$$\varphi(p)=-\tfrac{1}{4}\ln[1+2\lambda U(p)/\pi\hbar v_0]. \tag{9}$$

Using an earlier calculation[10] of $H^-(x,0)$ we obtain $H^+(x,0)$ by Hermitean conjugation and $H^-(x,t)$ by Eq. (4). Thus we have the essential operator part of Eq. (2):

$$\begin{aligned}&[H^+(x,0),H^-(x',t)]\\ &\quad=C(x'-x,t)[B^{-1\dagger}(x,0)B(x,0)A^\dagger(x,0)A^{-1}(x,0),B^\dagger(x',t)B^{-1}(x',t)A^{-1\dagger}(x',t)A(x',t)],\end{aligned} \tag{10}$$

where

$$C(x'-x,t)=L^{-2}\exp\{i[(k_{1F}-k_{2F})(x'-x)-v_0(k_{1F}+k_{2F})t]\}\exp(-2\alpha) \tag{11}$$

and[1]

$$\alpha=(2\pi/L)\sum_{p>0}p^{-1}(e^{2\varphi(p)}-1)=(2\pi/L)\sum_{p>0}[E^{-1}(p)-p^{-1}],$$

with

$$E(p)\equiv p[1+2\lambda U(p)/\pi\hbar v_0]^{1/2}. \tag{12}$$

The operators in (10) are

$$A(x,t)=\exp\{(2\pi/L)\sum_{p>0}p^{-1}\rho_1(-p)e^{ipx}e^{\varphi(p)}\exp[-iE(p)v_0t]\},$$

$$B(x,t)=\exp\{(2\pi/L)\sum_{p>0}p^{-1}\rho_2(p)e^{-ipx}e^{\varphi(p)}\exp[-iE(p)v_0t]\},\tag{13}$$

with Hermitean conjugates obtainable using the identity $\rho_i^\dagger(p)=\rho_i(-p)$. Averaging the exponentiated Bose-Einstein operators, one obtains for Eq. (2)

$$dj/dt=(2v_0/\hbar^2)L^{-2}\int dx\int dR\,\langle W^*(x)W(x+R)\rangle\exp[i(k_{1\mathrm{F}}-k_{2\mathrm{F}})R]$$
$$\times\int_{-\infty}^{\infty}dt(-2i)\sin[v_0(k_{1\mathrm{F}}+k_{2\mathrm{F}})t]\Sigma(v_0t+R)\,\Sigma(v_0t-R)\exp(Q_1-Q_1).\tag{14}$$

The calculations are greatly simplified if the scattering potential is spatially uncorrelated, i.e., $\langle W^*(x)W(x+R)\rangle=M\delta(R)$, where M is a constant. The remaining quantities are

$$\Sigma(y)\equiv\exp[(2\pi/L)\sum_{p>0}p^{-1}e^{ipy}]=\sum_{p>0}e^{ipy}\tag{15}$$

as previously defined,[1] and the covergent integrals

$$Q_1(R,t)=(4\pi/L)\sum_{p>0}\left(p^{-1}[1-\exp(iv_0pt)\cos pR]-E^{-1}(p)\{1-\exp[iE(p)v_0t]\cos pR\}\right),\tag{16}$$

$$Q_2(R,t)=(8\pi/L)\sum_{p>0}E^{-1}(p)f[E(p)]\{\sin^2\tfrac{1}{2}[pR+v_0E(p)t]\sin^2\tfrac{1}{2}[pR-v_0E(p)t]\},\tag{17}$$

with the Bose-Einstein distribution function being

$$f(E)=[\exp(v_0E/kT)-1]^{-1}.\tag{18}$$

To evaluate Q_1 and Q_2, essentially "Debye-Waller" factors, analytically we go to the weak-coupling limit and assume $U(p)$ is slowly varying up to a cutoff at $p=p_0$, so that $U(p)\approx U(0)$ for $p<p_0$ and vanishes for $p>p_0$. We then define the dimensionless coupling constant

$$g=2\lambda U(0)/\pi\hbar v_0\tag{19}$$

which is now assumed to be small, $|g|\ll 1$, and evaluate dj/dt to leading order in g. By our assumptions on the scattering mechanism, we can set $R=0$ in the integrals, which then assume the following asymptotic forms for large t:

$$\exp[Q_1(0,t)]=(1+p_0v_0|t|)^g,\tag{20}$$

$$\exp[-Q_2(0,t)]=\exp[-(\gamma kT|t|/\hbar)],\tag{21}$$

where γ is a number $O(1)$. Study of the integral in (14) shows that the dominant contributions are from regions where the complex phases vanish and for a range of t of the order of $\hbar/kT$. For small g, we take the slowly varying factor as given in (20) out of the integral, Eq. (14), replacing it by an order-of-magnitude estimate, $\exp[Q_1(0,\hbar/kT)]$. We can set $g=0$ in the remaining integrals, to leading order. Recognizing the remainder with $g=0$ as the decay rate of a current of noninteracting fermions, $(dj/dt)_0$, one easily proves that this quantity is proportional to the current itself, with a constant of proportionality $-\tau_0^{-1}$, where τ_0 is the scattering decay time for free particles. [The simplest method of proof involves an independent evaluation of Eq. (2) using the free-fermion operators suitable for the case $g=\lambda=0$.] Thus, we establish that the current decays exponentially,

$$\frac{dj}{dt}=\left(1+\frac{p_0v_0\hbar}{kT}\right)^g\left(\frac{dj}{dt}\right)_0=-\left(1+\frac{p_0v_0\hbar}{kT}\right)^g\tau_0^{-1}j\tag{22}$$

and obtain the new decay time by inspection. Identification of $p_0v_0\hbar$ with $k\theta$ and of the ratio of decay times to the ratio of conductivities completes the derivation of Eq. (1), constituting the principal result.

It is evident that if a BCS superconducting, coherent ground state is assumed for 1D electrons with attractive interactions, the fluctuations would be so large at finite temperature as to destroy the self-consistency of the assumed long-range order, and restore finite conductivity. As an example, a free-electron sea interacting with phonons has been shown to behave as a Peierls insulator for 1D at low temperature,[8] rather than as a superconductor as would be the case[9] in 3D. To examine this behavior within the context of our exactly soluble model, I have calculated the Cooper-pair amplitude, $\langle\psi_1(x)\psi_2(x')\rangle$, with the angular brackets indicating the matrix element between eigenstates of H differing by two particles in occupation number and thermally averaged. This quantity can also be obtained as the square root of the pair-pair correlation function obtained by Luther and Peschel,[2] evaluated at infinite separation. Calculating it directly, using the wave-operator representation of Ref. 1, one finds that the key to the behavior of this amplitude is dominated by two divergent integrals, denoted Q_3 and Q_4:

$$\langle\psi_1(x)\psi_2(x')\rangle \propto \exp(Q_3)\exp(-Q_4), \tag{23}$$

where

$$\exp(Q_3) = \exp[(2\pi/L)\sum_{p>0} p^{-1}(1-e^{-2\varphi(p)})]$$

$$\approx (1+Lp_0)^{-g/2} \tag{24}$$

Thus $\exp(Q_3)=0$ for repulsive two-body forces, $=1$ for $g=0$, and diverges weakly for attractive forces. Q_4 contains the temperature dependence. We have

$$\exp(-Q_4) = \exp\{-(4\pi/L)\sum_{p>0} p^{-1}f[E(p)]e^{-2\varphi(p)}\}$$

$$\approx \exp(-2LkT/\hbar v_0); \tag{25}$$

and thus, at any finite temperature, and regardless of the sign of magnitude of the two-body forces, $\exp(-Q_4)\equiv 0$. The vanishing of the Cooper-pair amplitude at any finite temperature is compatible only with a lack of long-range superconducting order, and confirms the finite conductivity obtained in Eq. (1). In conclusion, while attractive forces lead to many-body effects which can enhance the conductivity of 1D fermions, they cannot make them superconduct at finite temperature.

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